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The double-well phase of matrix models: the large-N general solution

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Abstract. We provide the general solution of the large-N limit of matrix models with even polynomial potential in the phase with two minima. The solution is given both in the saddle point and in the orthogonal polynomials approaches.

1. Introduction

In recent times a great interest has developed in the subject of the large-N limit of matrix models. They have been shown to provide exactly soluble models for statistical mechanics in two dimensions, being capable to describe random surfaces or spin systems on random lattices. The starting point may be traced back to Kazakov's work of 1986 [1], followed by many other papers (see for example [2] and references cited therein). However, in these investigations most of the attention has been devoted to the 'perturbative' sector, characterized by the presence of a single minimum in the even polynomial potential appearing in the partition function. On the other hand, the range of the parameters is wider and often involves the appearance of other regions, which classically correspond to new minima. In the large-N limit the transition to these sectors is non-analytical and gives rise, in the parameters' space, to a phase diagram. Typically the transition is of third order, and has been studied for the first time by Gross and Witten [3] for SU(N) models (see also Jurkiewicz and Zalewski [5]), and by Shimamune [4] and us [6] for models with Hermitian matrices.

Having this in mind, it seems useful to provide the full solution to the situation next to the perturbative one, corresponding to the double-well configuration of the general polynomial potential. The solution is found both in the saddle point approach, which is very effective for the investigation of the phase lines, and in the orthogonal polynomial approach, which is most commonly used in the recent literature. The connection between the two approaches is also explicitly given.

2. The saddle point method

Our starting point is the partition function for a Hermitian $N \times N$ matrix variable with a generic even polynomial potential

$$Z_N = C_N \int \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1 \leqslant i < j \leqslant N} \mathrm{d}^2 M_{ij} \exp\left(-\mathrm{Tr} \sum_{k=0}^{n} \frac{c_k}{k+1} \frac{1}{N^k} M^{2k+2} \right). \quad (1)$$

The matrix is diagonalized and in a standard way one obtains an expression involving the eigenvalues. All the inessential factors, like the volume of the unitary group, are absorbed into the coefficient C_N so that

$$Z_N = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta^2(\lambda_1, \ldots, \lambda_N) \exp\left(-N \sum_{i=1}^{N} V(\lambda_i)\right). \tag{2}$$

The Jacobian of the transformation is the square of the Vandermonde determinant

$$\Delta(\lambda_1, \ldots, \lambda_N) = \prod_{i<j} (\lambda_i - \lambda_j) = (-1)^{N(N-1)/2} \det[\lambda_i^k]_{i=1,N}^{k=0,N-1} \tag{3}$$

and

$$V(\lambda) = \sum_{k=0}^{n} \frac{c_k}{k+1} \lambda^{2k+2}. \tag{4}$$

In the large-N limit, the ordered eigenvalues are distributed according to a density function that generalizes Wigner's semicircle law (the Gaussian case). The density $u(\lambda)$, with support L, satisfies a saddle point equation:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} V(\lambda) = P \int_L \frac{u(\mu)}{\lambda - \mu} \mathrm{d}\mu \qquad \lambda \in L \tag{5}$$

together with a normalization condition and the requirement of vanishing at the endpoints of the suitably chosen support L. For the perturbative phase, where V has a single minimum in $\lambda = 0$, one sets $L = [-2a, 2a]$ and the solution has been given in the paper [7]. It is a symmetric function of the type $u(\lambda) = (1/\pi)\sqrt{4a^2 - \lambda^2}\, P(\lambda)$, where P is a polynomial. The vanishing in $\lambda = 0$ gives rise to a new phase for which it is necessary to split the support into $L = [-B, -A] \cup [A, B]$. This situation corresponds to a double-well configuration of the potential, as will be better shown in the approach with orthogonal polynomials.

It is useful to define two related parameters:

$$s = \tfrac{1}{2}(B^2 + A^2) \qquad d = \tfrac{1}{2}(B^2 - A^2). \tag{6}$$

The solution to the integral equation is accordingly searched in the form

$$u(\lambda) = \frac{1}{\pi} |\lambda| \sqrt{(B^2 - \lambda^2)(\lambda^2 - A^2)} \sum_{r=0}^{n} p_r \lambda^{2r}. \tag{7}$$

The square root factor is peculiar of the inversion of (5) for the given type of interval, by means of the Poincaré–Bertrand formula. Placing this ansatz into equation (5) one has to perform integrals of the type

$$I_k = \frac{1}{2\pi} P \int_{A^2}^{B^2} \mathrm{d}x \frac{x^k}{x - \xi} \sqrt{(B^2 - x)(x - A^2)} \tag{8}$$

which are easily seen to satisfy the recurrence relation $I_{k+1} = \xi I_k + \alpha_k$, where

$$\begin{aligned}\alpha_k &= \frac{1}{2\pi} \int_{A^2}^{B^2} \mathrm{d}x \, x^k \sqrt{(B^2 - x)(x - A^2)} \\ &= \frac{1}{4} d^2 s^k {}_2F_1\left(-\frac{k}{2}, \frac{1-k}{2}; 2; \frac{d^2}{s^2}\right) = \frac{1}{4} d^2 (s+d)^k {}_2F_1\left(-k, \frac{3}{2}; 3; \frac{2d}{s+d}\right).\end{aligned} \tag{9}$$

The last equality corresponds to a quadratic transformation ([8] vol 1, p 112, equation (17)). The recurrence relation yields an expression for the integral (8):

$$I_k = \xi^k I_0 + \sum_{r=0}^{k-1} \alpha_{k-1-r}\xi^r. \tag{10}$$

Equation (5) then gives the following relation for the coefficients p_r in the density of eigenvalues:

$$\sum_{k=0}^{n} c_k \lambda^{2k} = \sum_{r=0}^{n-1} p_r\left(\lambda^{2r+2} - s\lambda^{2r} - 2\sum_{j=0}^{r-1} \alpha_{r-1-j}\lambda^{2j}\right) \tag{11}$$

whence

$$c_k = p_{k-1} - sp_k - 2\sum_{r=0}^{n-k-2} \alpha_r p_{k+r+1}. \tag{12}$$

The inspection of (12) in the cases $k = 1, 2, \ldots$ suggests a general solution of the kind

$$p_r = \sum_{j=0}^{n-r-1} \beta_j c_{r+j+1} \tag{13}$$

together with a constraint for the endpoints of L

$$\sum_{r=0}^{n} \beta_r c_r = 0 \tag{14}$$

where

$$\beta_r = s^r {}_2F_1\left(-\frac{r}{2}, \frac{1-r}{2}; 1; \frac{d^2}{s^2}\right) = (s+d)^r {}_2F_1\left(-r, \frac{1}{2}; 1; \frac{2d}{s+d}\right). \tag{15}$$

To prove equation (13) we notice the relations

$$s\beta_{r-1} + 2(r-1)\alpha_{r-2} = \beta_r \qquad (r \geqslant 1) \tag{16}$$

$$\sum_{r=0}^{n} \alpha_r \beta_{n-r} = (n+1)\alpha_n \tag{17}$$

the first of which is trivial, while the second follows from the identity

$$\phi_1\phi_2 = \left(t\frac{\partial}{\partial t} + 1\right)\phi_2 \tag{18}$$

where

$$\phi_1(t, z) = \sum_{r=0}^{\infty} t^r {}_2F_1(-r, \tfrac{1}{2}; 1; z) = (1-t)^{-1/2}(1-t+zt)^{-1/2} \tag{19}$$

$$\phi_2(t, z) = \sum_{r=0}^{\infty} t^r {}_2F_1(-r, \tfrac{3}{2}; 3; z) = 4(\sqrt{1-t} + \sqrt{1-t+zt})^{-2}. \tag{19'}$$

(The infinite series are easily evaluated by expressing the hypergeometric functions as Eulerian integrals.) Next, by direct insertion of (13) into (12), the right-hand side becomes

$$\begin{aligned} p_{k-1} - sp_k - 2\sum_{r=0}^{n-k-2} \alpha_r p_{k+r+1} \\ = \sum_{j=0}^{n-k} \beta_j c_{k+j} - s\sum_{j=0}^{n-k-1} \beta_j c_{k+j+1} - 2\sum_{r=0}^{n-k-2} \alpha_r \sum_{j=0}^{n-k-r-2} \beta_j c_{k+r+j+2}. \end{aligned} \tag{20}$$

In the last term the sums are exchanged and use is made of equation (17), to obtain

$$\sum_{j=0}^{n-k} \beta_j c_{k+j} - s \sum_{j=0}^{n-k-1} \beta_j c_{k+j+1} - 2 \sum_{j=0}^{n-k-1} c_{k+j+1} j\alpha_{j-1}$$

$$= c_k + \sum_{j=0}^{n-k-1} c_{k+j+1}(\beta_{j+1} - s\beta_j - 2j\alpha_{j-1}) = c_k \tag{21}$$

by virtue of equation (16). This concludes our proof.

Finally, by inserting the explicit expression for the density in terms of s and d into the normalization condition

$$2\int_A^B u(\lambda)\,\mathrm{d}\lambda = 1 \tag{22}$$

we get the second equation for the endpoints of the support L:

$$2\sum_{r=1}^{n} r c_r \alpha_{r-1} = 1. \tag{23}$$

The two equations (14) and (23) provide a system for the evaluation of the unknowns s and d, that define the support. In particular, the solution with $A=0$, equivalent to $d=s$, corresponds to the merging into the single-segment solution. It implies a relation between the parameters of the model (the coefficients in the potential) leading in general to a surface in the parameters' space dividing the perturbative phase, with eigenvalues in a single interval, from the 'double-well phase'. As an example, the sextic case ($n=2$) is investigated in [9, 10]. Other phases may also be present, for those values of the parameters such that either the perturbative or the double well densities would become negative on their support. The investigation of these more complicated phases would follow the same approach as described here.

3. The orthogonal polynomials method

A different and powerful approach to the evaluation of the partition function Z_N employs a set of suitably chosen orthogonal polynomials [11]. The method exploits the invariance properties of determinants: $\det[\lambda_i^k] = \det[P_k(\lambda_i)]$, where $P_k(\lambda)$ is an arbitrary polynomial of order k, with the coefficient of λ^k equal to one. If moreover the polynomials are required to satisfy the orthogonality relation

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, P_n(\lambda) P_m(\lambda)\, \mathrm{e}^{-NV(\lambda)} = \delta_{nm} h_n \tag{24}$$

all the integrals in the partition function may be performed, leaving us with a very simple result for the free energy for arbitrary N:

$$E_N = -\frac{1}{N^2}\log Z_N = -\frac{1}{N}\log h_0 - \frac{1}{N}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\log R_k \tag{25}$$

where

$$R_k = \frac{h_k}{h_{k-1}} \geqslant 0. \tag{26}$$

The polynomials have a definite parity and satisfy the recursive relation, with $R_0 = 0$,

$$P_{n+1}(\lambda) = \lambda P_n(\lambda) - R_n P_{n-1}(\lambda) \tag{27}$$

where the coefficients R_k also solve a complicated recursive equation. This equation follows from the explicit computation, using (24) and (27), of the integrals in the equality

$$
\begin{aligned}
ih_{i-1} &= \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} P_i(\lambda)\right) P_{i-1}(\lambda) \\
&= 2N \sum_{r=0}^{n} c_r \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)} \lambda^{2r+1} P_i(\lambda) P_{i-1}(\lambda).
\end{aligned} \tag{28}
$$

We shall derive this equation in the large-N limit; the procedure strictly depends on the structure of the potential [12]. Indeed, the low coefficients R_1, R_2, etc, may be computed directly and show in the large-N limit a behaviour which depends on the position of the minima of the potential. For example,

$$
R_1 = \frac{h_1}{h_0} = \frac{\int \mathrm{d}\lambda\, \lambda^2 \exp[-NV(\lambda)]}{\int \mathrm{d}\lambda\, \exp[-NV(\lambda)]} \tag{29}
$$

in the large-N limit has value zero in the perturbative phase and value $\Lambda^2 = \lambda^2_{\min}$ in the double-well phase. In the perturbative phase one may therefore interpolate all the coefficients R_k by a single function $R(x)$ with boundary value $R(0) = 0$. In the double-well phase one is forced to define two distinct interpolating functions $f(x)$ or $g(x)$, with $x = i/N \in [0, 1]$, such that $R_{2i} = f(2i/N)$ and $R_{2i-1} = g((2i-1)/N)$. They satisfy different boundary conditions: $f(0) = 0$ and $g(0) = \Lambda^2$. It is useful to define the auxiliary functions $u = f + g$, $v = fg$.

To obtain the equations for u and v we start from (27) in terms of f or g, and then iterate it. For a finite number of iterations, as required by the polynomial potential, one has to care only about parity, and in the large-N limit $f = f(i/N) = f((i+2)/N) = \ldots$, $g = g((i+1)/N) = g((i+3)/N), \ldots$

$$
\lambda P_i(\lambda) = P_{i+1}(\lambda) + f P_{i-1}(\lambda) \tag{30}
$$

$$
\lambda^2 P_i(\lambda) = P_{i+2}(\lambda) + u P_i(\lambda) + v P_{i-2}(\lambda). \tag{31}
$$

It is easy to see that we may write in general:

$$
\lambda^{2r+1} P_i(\lambda) = \sum_{j=0}^{2r} \gamma_{j,r} [P_{i+1+2(r-j)}(\lambda) + f P_{i-1+2(r-j)}(\lambda)] \tag{32}
$$

whence iterating once:

$$
\begin{aligned}
\lambda^{2r+2} P_i(\lambda) &= \sum_{j=0}^{2r} \gamma_{j,r} [P_{i+2+2(r-j)}(\lambda) + u P_{i+2(r-j)}(\lambda) + v P_{i-2+2(r-j)}(\lambda)] \\
&= \sum_{j=0}^{2r+2} \gamma_{j,r+1} P_{i+2(r+1-j)}(\lambda).
\end{aligned} \tag{33}
$$

The comparison of the two right-hand sides gives the recursive relation:

$$
\gamma_{j,r+1} = \gamma_{j,r} + u\gamma_{j-1,r} + v\gamma_{j-2,r}. \tag{34}
$$

By defining the generating function

$$
\phi(x, y) = \sum_{j,r=0}^{\infty} \gamma_{j,r} x^j y^r \tag{35}
$$

one rewrites the recursive relation into an algebraic equation

$$\frac{1}{y}(\phi - 1) = (1 + ux + vx^2)\phi \tag{36}$$

with solution

$$\begin{aligned}\phi = \frac{1}{1 - y(1 + ux + vx^2)} &= \sum_{r=0}^{\infty} y^r (1 + ux + vx^2)^r \\ &= \sum_{r=0}^{\infty} y^r \sum_{j=0}^{\infty} (-x)^j v^{j/2} C_j^{-r}\left(\frac{u}{2\sqrt{v}}\right)\end{aligned} \tag{37}$$

where $C_j^{\nu}(x)$ are Gegenbauer's polynomials. The solution is therefore

$$\gamma_{j,r} = (-1)^j v^{j/2} C_j^{-r}\left(\frac{u}{2\sqrt{v}}\right). \tag{38}$$

If we now insert the relations (compare with (32))

$$\begin{aligned}\lambda^{2r+1} P_i(\lambda) &= \sum_{j=0}^{2r+1} (\gamma_{j,r} + f\gamma_{j-1,r}) P_{i+1+2(r-j)}(\lambda) \\ \lambda^{2r+1} P_{i-1}(\lambda) &= \sum_{j=0}^{2r+1} (\gamma_{j,r} + g\gamma_{j-1,r}) P_{i+2(r-j)}(\lambda)\end{aligned} \tag{39}$$

into the duplicated relation (28) and perform the integrals, we obtain

$$\begin{aligned}x &= 2f \sum_{r=0}^{n} c_r(\gamma_{r,r} + g\gamma_{r-1,r}) \\ x &= 2g \sum_{r=0}^{n} c_r(\gamma_{r,r} + f\gamma_{r-1,r}).\end{aligned} \tag{40}$$

Adding and subtracting, we finally get the equations for $u(x)$ and $v(x)$

$$x = 2v \sum_{r=1}^{n} c_r \gamma_{r-1,r} \tag{41}$$

$$0 = \sum_{r=0}^{n} c_r \gamma_{r,r}. \tag{42}$$

To establish the connection with the saddle point method, we note that the explicit expressions ([8] vol 2, p 176, equations (21) and (22))

$$\gamma_{2j,r}(u, v) = \frac{r!}{j!(r-j)!} v^j {}_2F_1\left(-j, j-r; \frac{1}{2}; \frac{u^2}{4v}\right) \tag{43}$$

$$\gamma_{2j+1,r}(u, v) = \frac{r!}{j!(r-j-1)!} uv^j {}_2F_1\left(-j, j-r+1; \frac{3}{2}; \frac{u^2}{4v}\right) \tag{44}$$

imply the following identifications:

$$\frac{d^2}{4}\gamma_{r-1,r}\left(s, \frac{d^2}{4}\right) = r\alpha_{r-1} \qquad \gamma_{r,r}\left(s, \frac{d^2}{4}\right) = \beta_r. \tag{45}$$

It turns out that for the special value $x = 1$ the two equations (41) and (42) are respectively equivalent to (23) and (14), provided we identify $u(1) = s$ and $v(1) = d^2/4$.

Finally, we remark that in the double-well phase, the large-N limit of the free energy is given by

$$
\begin{aligned}
E_\infty &= V(\Lambda) - \frac{1}{2}\int_0^1 (1-x)\log v(x)\,\mathrm{d}x \\
&= V(\Lambda) - \frac{1}{4}\log v(1) + \frac{1}{2}\int_0^{v(1)} \left(x(v) - \frac{x(v)^2}{2}\right)\frac{\mathrm{d}v}{v}.
\end{aligned}
\tag{46}
$$

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